SPECTRA OF SUBMERSIONS
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Let \((M, g), (B, j)\) be two connected compact Riemannian manifolds without boundary and let \(\pi : M \rightarrow B\) be a submersion. We investigate the relations between the spectra of the Laplace-Beltrami operators acting on functions defined respectively on \(M, B\) and on the fibers \(F_b = \pi^{-1}(b), b \in B\).

The problem was completely solved, via representation theory, by G. Besson and me (see [1]) when the submersion is Riemannian and the fibers are totally geodesic submanifolds of \(M\). Namely, we gave an explicit method to compute the eigenvalues with multiplicities and the eigenfunctions of \(\Delta_M\) by the ones of \(\Delta_F\) (in this case all the fibers are isometric to a typical fiber \(F\)) and the ones of the horizontal Laplacian.

When the submersion is Riemannian and the fibers are minimal submanifolds of \(M\), one gets (see [2]) the following estimates: for any integer \(N\), the spectra of \(\Delta_M\) and \(\Delta_B\) are related by

\[
\lambda_N(\Delta_M) \geq \frac{1}{8(p+1)^2}\lambda_{k+1}(\Delta_B)
\]

\[
\sum_{i=1}^{N} \lambda_i(\Delta_M) \geq \frac{1}{2} \sum_{j=1}^{k} \lambda_j(\Delta_B) + \frac{k}{8(p+1)}\lambda_{k+1}(\Delta_B)
\]

where \(p\) is the rank of the subspace spanned by the \(N\) first eigenfunction of \(\Delta_M\) and where \(k\) is the integer part of \(\frac{N}{p+1}\). Moreover, we prove that the resolvent operator \((\Delta_M + \lambda)^{-1}\) and the heat operator \(e^{-t\Delta_M}\) dominate \((\Delta_B + \lambda)^{-1}\) and \(e^{-t\Delta_B}\) respectively (see [3]).

I finish with some recent results concerning Riemannian submersions with basic mean curvature vector field and almost-Riemannian submersions, see [4].

References

